Lec 03 No. Date. Jun 25, 2026

Review:

Derivative Rules

1° $(f+g)'(x) = f'(x)+g'(x)$

2° $(cf)'(x) = cf'(x)$

3° $\frac{d}{dx}x^n = nx^{n-1}$

4° $(f\circ g)'(x) = \frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$

5° $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

6° $(fg)'(x) = f'(x)g(x)+f(x)g'(x)$

Derivatives of inverse function:

★ $f^{-1}(x)$ is the inverse function of $f(x)$ if

$f(f^{-1}(x)) = x$ & $f^{-1}(f(x)) = x$

★Notation: for trig functions $\sin^{-1}(x)$ we use $\arcsin(x)$

similarly

$\cos^{-1}(x) \longrightarrow \arccos(x)$

$\tan^{-1}(x) \longrightarrow \arctan(x)$

$\sec^{-1}(x) \longrightarrow \operatorname{arcsec}(x)$

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example: $\frac{d}{dx} \arctan(x) = ?$

$\tan(\arctan(x)) = x$

we differentiate both sides:

$\frac{d}{dx} \tan(\arctan(x)) = \frac{d}{dx} x = 1$

$\rightarrow$ we know: $\frac{d}{dt} \tan(t) = \sec^2(t)$ therefore:

$\Rightarrow \sec^2(\arctan(x)) \cdot \frac{d}{dx} \arctan(x)$ [chain Rule]

$\Rightarrow \frac{d}{dx} \arctan(x) = \frac{1}{\sec^2(\arctan(x))}$

$\rightarrow$ we know: $\tan^2 = \sec^2(x) - 1$ therefore:

$\Rightarrow \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + [\tan(\arctan(x))]^2}$

$\Rightarrow \boxed{\frac{1}{1+x^2}}$

example: $\frac{d}{dx} \ln(x) = ?$ if $\frac{d}{dt} e^t = e^t$

$\rightarrow \frac{d}{dx} e^{\ln(x)} = \frac{d}{dx} x = 1 \Rightarrow e^{\ln(x)} \cdot \frac{d}{dx} \ln(x) \Rightarrow$

$\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$

we know:

$\ln(e^x) = x^2$

$e^{\ln(x)} = x$

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★ Remember
$$\begin{cases} a^x = (e^{\ln(a)})^x = e^{x.\ln(a)} \\ \log a^{(x)} = \dfrac{\ln(x)}{\ln(a)} \end{cases}$$

~~★ example~~

Logarithmic Differentiation :

$$y = \frac{(x^2-1)(x-3)}{(x^3+1)(x+2)}$$

★ we can use the derivative rules to solve but there's another way!

→ we take the log of both sides :

$$\ln(y) = \ln\left(\frac{(x^2-1)(x-3)}{(x^3+1)(x+2)}\right)$$

★ logs convert multiplication into additions! & more!

for example : $\ln(ab) = \ln(a) + \ln(b)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$\ln(b^a) = a\ln(b)$

Therefore :

$$\ln(y) = \ln((x^2-1)(x-3)) - ((x^3+1)(x+2)) \Rightarrow$$

$$[\ln(x^2-1) + \ln(x-3)] - [\ln(x^3+1) + \ln(x+2)] \Rightarrow$$

$$\ln(x-1) + \ln(x+1) + \ln(x-3) - \ln((x+1)(x^2-x+1)) - \ln(x+2)$$

$$\Rightarrow \ln(x-1) + \cancel{\ln(x+1)} + \ln(x-3) - \cancel{\ln(x+1)} - \ln(x^2-x+1) - \ln(x+2)$$

$$\Rightarrow \frac{d}{dx}\ln(y) = \frac{d}{dx}(\ln(x-1) + \ln(x-3) - \ln(x^2-x+1) - \ln(x+2))$$

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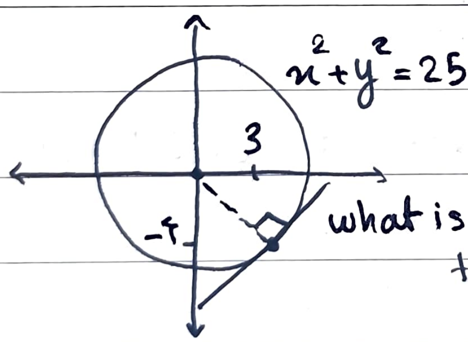
$$\Rightarrow \frac{1}{y} = \frac{1}{x-1} + \frac{1}{x-3} - \frac{1}{x^2-x+1}(2x-1) - \frac{1}{x+2} \Rightarrow$$

$$\frac{dy}{dx} = y\left[\frac{1}{x-1} + \frac{1}{x-3} + \frac{2x-1}{x^2-x+1} - \frac{1}{x+2}\right] \Rightarrow$$

$$\boxed{\frac{(x^2-1)(x-3)}{(x^3+1)(x+2)}\left[\frac{1}{x-1} + \frac{1}{x-3} + \frac{2x-1}{x^2-x+1} - \frac{1}{x+2}\right]}$$

Implicit Differentiation:

$$x^2 + y^2 = 25$$ (circle: center at $(0,0)$ & radius 5)



what is the slope of the $(3,-4)$ tangent line?

$$y^2 = 25 - x^2 \Rightarrow y = \pm\sqrt{25-x^2}$$

+ upper semi-circle
− lower semi-circle

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{d}{dx}\left(-\sqrt{25-x^2}\right) \Rightarrow \frac{d}{dx}\left(-(25-x^2)^{\frac{1}{2}}\right) \Rightarrow$$

$$-\frac{1}{2}(25-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(25-x^2) \Rightarrow$$

$$\frac{x}{\sqrt{25-x^2}} \Rightarrow$$ evaluating it at $x=3$ $\Rightarrow$

*we can do this easier with implicit differentiation $\Rightarrow$

$$\frac{3}{\sqrt{25-9}} = \frac{3}{\sqrt{16}} = \boxed{\frac{3}{4}}$$

$$\Rightarrow x^2 + y^2 = 25$$

$$\Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) = 0$$

$$\Rightarrow 2x + 2y \cdot \frac{d}{dx} \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow$$

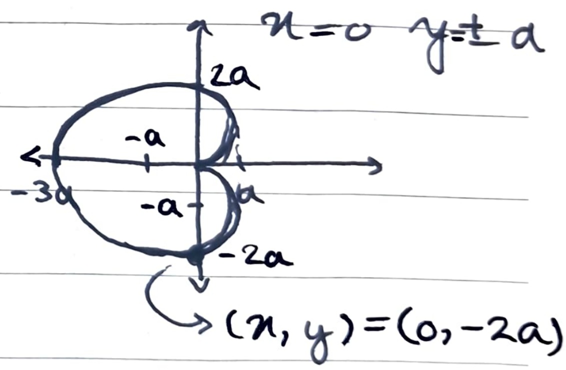
$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,-4)} = \frac{-3}{-4} = \boxed{\frac{3}{4}}$$

★Challenging example :

Cardioid with parameter equation:

$$(x^2+y^2)^2 + 4ax(x^2+y^2) - 4a^2y^2 = 0$$

take $\frac{d}{dx}$ on both sides:



$$2(x^2+y^2)(2x+2y\frac{dy}{dx}) + 4a(x^2+y^2) + 4ax(2x+2y\frac{dy}{dx}) \Rightarrow$$

$$4x(x^2+y^2) + 4y(x^2+y^2)\cdot\frac{dy}{dx} + 4a(x^2+y^2) + 8ax^2 + 8axy\frac{dy}{dx} - \dots$$

$$\dots 8a^2y\cdot\frac{dy}{dx} = 0 \Rightarrow$$

$$\frac{dy}{dx} = \frac{-4x(x^2+y^2) - 4a(x^2+y^2) - 8ax^2}{4y(x^2+y^2) + 8axy - 8a^2y} \Rightarrow$$

now we plug in $x=0$ $y=-2$

$$\Rightarrow \frac{-16a}{-32a^3+16a^3} = \frac{1}{a} \text{ ?!}$$

→ we did something wrong somewhere as the answer should be 0

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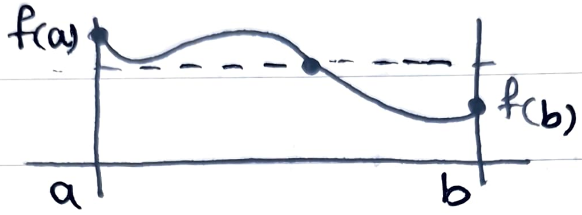
Intermediate Value Thm:

$f(x)$ is continuous on $[a, b]$ ↗ include end points

⇒ $f(x)$ takes on every value between $f(a)$ & $f(b)$ somewhere between $x = a$ & $x = b$

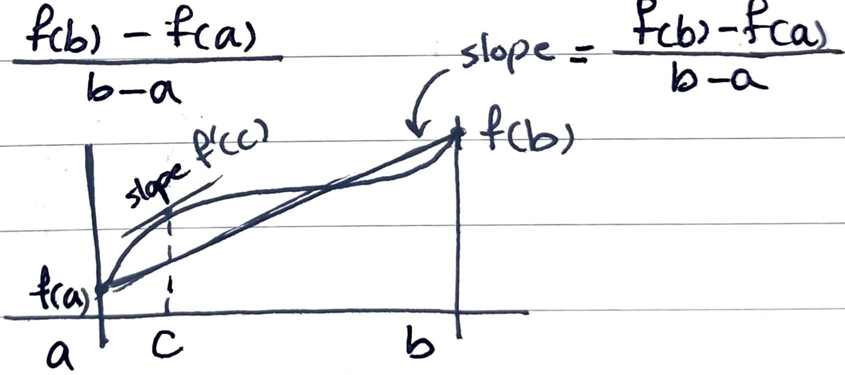


Mean Value Thm:

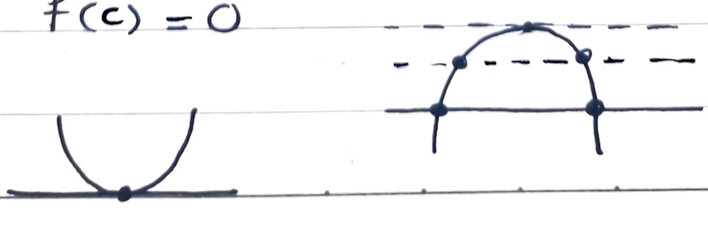
Suppose $f(x)$ is continuous on $[a,b]$ & differentiatable on (ab).

Then there is a point $c \in (a,b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Fact: If $f(x)$ is differentiable & has a (local) maximum or minimum at c, then $f'(c) = 0$



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what are the local maxima & minima (if any) of $y=x^3-x$?

→ look for values of x where $\frac{dy}{dx}=0$

$\frac{dy}{dx}$ $3x^2-1=0$ when $x=\frac{\pm 1}{\sqrt{3}}$ candidates $x=\frac{1}{\sqrt{3}}$, $x=-\frac{1}{\sqrt{3}}$

x	$(-\infty,-\frac{1}{3})$	$(-\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}},+\infty)$
$f'(x)$	+	0	−	0	+
$f(x)$	↑	local max	↓	local min	↑

*example : local max & min of

$$y=\frac{1}{1+x^2}$$

$$\frac{dy}{dx}=\frac{(1+x^2)-1(2x)}{(1+x^2)^2}=\frac{-2x}{(1+x^2)^2}=0 \text{ when } x=0$$

x	$(-\infty,0)$	0	$(0,+\infty)$
$f'(x)$	+	0	−
$f(x)$	↑	local max	↓

$$\lim_{x\to\infty}\frac{1}{1+x^2}=0^+$$

$$\lim_{x\to-\infty}\frac{1}{1+x^2}=0^+$$